

Non-ergodicity through quantum feedback

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Abstract. Open quantum systems usually reach a unique stationary state with ergodic dynamics. In other words, the ensemble averages and the time averages of the expectation values of open quantum systems are usually the same for all quantum trajectories. Although open quantum systems are in general ergodic, many classical stochastic processes are not. Hence if classical physics emerges from microscopic quantum models, there have to be mechanisms which induce non-ergodicity in open quantum systems. In this paper, we identify such a mechanism by showing that quantum feedback dramatically alters the dynamics of open quantum systems, thereby possibly inducing non-ergodicity and a persistent dependence of ensemble averages on initial conditions. As a concrete example, we study an optical cavity inside an instantaneous quantum feedback loop.

1. Introduction

Suppose a large number of identical physical systems generate time-dependent stochastic signals. The dynamics of these systems is called ergodic when any single, sufficiently long sample of the process has the same statistical properties as the entire process. More concretely, suppose we consider N identical systems with stochastic dynamics and

$$E(A) = \lim_{T \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N E_n(A, T) \quad (1)$$

denotes the ensemble average of the expectation values $E_n(A, T)$ of an observable A after a long time T . Then the system dynamics are ergodic when

$$E(A) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt E_n(A, t) \quad (2)$$

for all observables A and for all systems n . In other words, the system dynamics are ergodic when ensemble averages and time averages are the same for all observables and for all possible realisations of the process [1].

Standard examples of physical systems with ergodic dynamics are systems that eventually lose any information about their initial state. This applies, for example, to systems whose dynamics result in a unique stationary state [1]. In this case, the right hand sides of Eqs. (1) and (2) both sum over stationary state expectation values and are hence equal. In statistical physics, systems usually reach a unique thermal equilibrium that is independent of their initial state. Ergodic systems therefore lie at the heart of statistical physics [2, 3]. Nevertheless, an ergodic hypothesis remains hard to deduce from microscopic equations of motion [4, 5, 6]. Moreover, physical systems whose dynamics depend forever on their initial state are in general

non-ergodic. A particular class of non-ergodic systems that receive a lot of attention in the literature are physical systems with chaotic trajectories (see e.g. Refs. [7, 8]). For these, the right hand sides of Eqs. (1) and (2) are in general different.

In this paper, we are especially interested in the ergodicity of open quantum systems with Markovian dynamics [9]. The ensemble averages of the expectation values of these systems can be deduced from their density matrix ρ , which evolves according to a master equation in Lindblad form [10, 11],

$$\dot{\rho} = \mathcal{L}(\rho), \quad (3)$$

where \mathcal{L} denotes a superoperator. For example, if an open quantum system possesses only a single decay channel, then $\mathcal{L}(\rho)$ can be written as

$$\mathcal{L}(\rho) = -\frac{i}{\hbar} [H, \rho] + \frac{1}{2} \Gamma \left(2L\rho L^\dagger - [L^\dagger L, \rho]_+ \right), \quad (4)$$

where H is the system Hamiltonian, Γ denotes its spontaneous decay rate and L is a so-called Lindblad operator. However, to decide whether an open quantum system is ergodic or not, we also need to have a closer look at its individual quantum trajectories [12, 13, 14]. To do so, we need to unravel the above master equation in a physically meaningful way into equations that predict the dynamics of the individual quantum trajectories. In the case of spontaneous photon emission by a quantum optical system, a physically meaningful unravelling of Eq. (3) is obtained when writing $\mathcal{L}(\rho)$ in Eq. (4) as

$$\mathcal{L}(\rho) = -\frac{i}{\hbar} [H_{\text{cond}}, \rho] + \Gamma L\rho L^\dagger, \quad (5)$$

with the non-Hermitian conditional Hamiltonian H_{cond} given by

$$H_{\text{cond}} = H - \frac{i}{2} \hbar \Gamma L^\dagger L. \quad (6)$$

There are now two main terms contributing to the time evolution of ρ . The first term in Eq. (5) describes the dynamics of the open quantum system under the condition of no photon emission. In this case, it evolves according to a Schrödinger equation but with the system Hamiltonian H replaced by H_{cond} . The second term describes the effect of a photon emission. Up to normalisation, the state vector changes from $|\psi_n(t)\rangle$ into $L|\psi_n(t)\rangle$ in this case. The probability density for an emission to occur equals

$$I(t) = \Gamma \langle L^\dagger L \rangle_t, \quad (7)$$

where $\langle L^\dagger L \rangle_t$ denotes a time-dependent expectation value. Knowing H , Γ and L of a quantum optical system with spontaneous photon emission allows us to generate all its possible quantum trajectories.

Although it is widely believed that open quantum systems with Markovian dynamics are almost always ergodic, a general proof of their ergodic dynamics cannot be found in the literature [15]. What has been shown, for example, is that the dynamics of an open Markovian quantum system are ergodic if the system reaches a steady state that is independent of its initial state [9]. Non-ergodicity seems to require the existence of multiple stationary states (see e.g. Refs. [16, 17, 18]), which occur only in specially designed circumstances. This is usually not the case. However, many classical stochastic processes are non-ergodic and exhibit complex dynamics. Exactly how the complex dynamics of classical systems, including non-ergodicity and

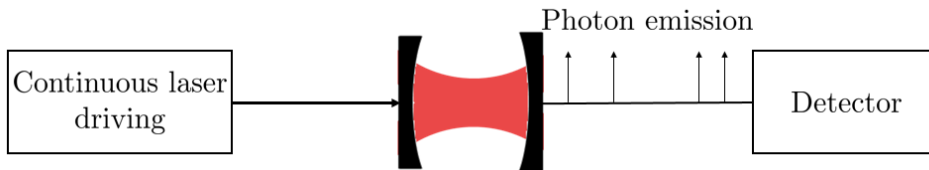


Figure 1. [Colour online] Schematic view of an optical cavity with continuous laser driving and spontaneous photon emission. A detector observes the field outside the resonator and registers the arrival of single photons at random times.

chaos, may arise in open quantum systems and in closed many-body quantum systems is the subject of extensive research [5, 6, 7, 19, 20, 21, 22].

In this paper, we identify a mechanism with the ability to induce complex dynamics in open quantum systems. It is shown that quantum feedback [23, 24, 25, 26] can induce non-ergodicity even when the quantum system possesses only a single unique stationary state. As we shall see below, the stationary state of an open quantum system inside a quantum feedback loop can become a repulsive fixed point of the system dynamics. When this applies, a system in its stationary state drifts away, even in the case of only tiny fluctuations. Instead of losing any information about the initial state, there can be a persistent dependence of the ensemble averages of expectation values on initial conditions. To show that this is indeed the case, we study a concrete example, namely the dynamics of an optical cavity with spontaneous photon emission inside an instantaneous quantum feedback loop. The quantum feedback-induced non-ergodicity of such cavities has already been shown to have applications in quantum-enhanced metrology [27].

There are five sections in this paper. In Section 2, we discuss how to model an optical cavity with spontaneous photon emission in a variety of situations, thereby providing the theoretical background for our work. In Section 3, we review the dynamics of an optical cavity when subject to continuous laser driving to provide a reference point for later discussions. As we shall see below, the dynamics of the resonator is linear and ergodic in this case. Afterwards, in Section 4, we replace the continuous laser driving by instantaneous quantum feedback and show that this relatively simple change results in significant changes of the system's dynamics. Finally, in Section 5 we review our findings.

2. Theoretical background

In this section, we introduce the theoretical tools needed to analyse the dynamics of an optical cavity with continuous laser driving and an optical cavity inside an instantaneous quantum feedback loop, like the ones shown in Figs. 1 and 2. When allowing for the leakage of photons through the cavity mirrors, the density matrix ρ of both systems obeys a master equation in Lindblad form, like the one in Eq. (4), which can be unravelled into individual quantum trajectories. Our starting point for the derivation of these equations is the total Hamiltonian H of the cavity and its surrounding free radiation field. This Hamiltonian consists essentially of four terms [27],

$$H = H_S + H_B + H_{SB} + H_{\text{laser}}. \quad (8)$$

The first two terms describe the free energy of the cavity field and the free radiation field surrounding the cavity, while the third term describes the system-bath interaction and the fourth term takes laser driving into account. In the following, we denote the energy of a single

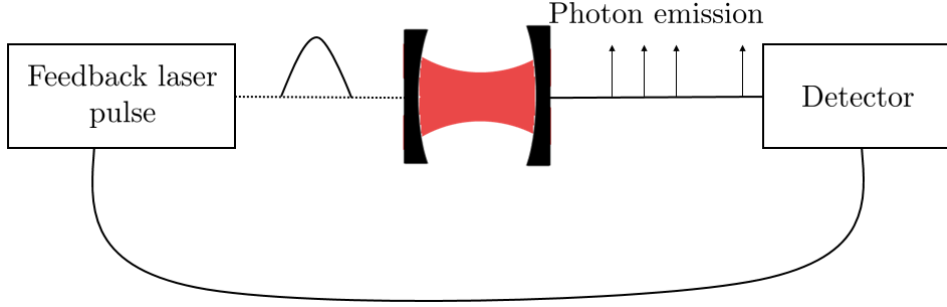


Figure 2. [Colour online] Schematic view of an optical cavity inside an instantaneous quantum feedback loop. Now the continuous laser driving is replaced by a random sequence of pulses. These are triggered by the detection of a single photon at the detector and displace the field inside the resonator in a well defined way.

photon by $\hbar\omega_{\text{cav}}$ and c and c^\dagger are its annihilation and creation operators respectively, with $[c, c^\dagger] = 1$. Moreover, $a_{\mathbf{k}\lambda}$ denotes the annihilation operator of a single photon with frequency ω_k , wave vector \mathbf{k} and polarisation λ outside the resonator, while $a_{\mathbf{k}\lambda}^\dagger$ denotes the corresponding creation operator, with $[a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}^\dagger] = \delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'}$. Using this notation,

$$H_S = \hbar\omega_{\text{cav}} c^\dagger c, \quad H_B = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda}. \quad (9)$$

Denoting the constants for the coupling between the cavity and the free radiation field by $g_{\mathbf{k}\lambda}$, the interaction Hamiltonian H_{SB} equals

$$H_{\text{SB}} = \sum_{\mathbf{k}\lambda} \hbar g_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\dagger c + \text{H.c.} \quad (10)$$

after applying the usual rotating wave approximation. Finally, the laser Hamiltonian, which describes the external driving of the cavity field, can be written as

$$H_{\text{laser I}} = \frac{1}{2} \hbar \Omega (c + c^\dagger) \quad (11)$$

within the above mentioned approximation and in the interaction picture with respect to $H_0 = H_S$. Here Ω denotes the Rabi frequency of the applied laser field. Without loss of generality, we can assume that Ω is real; potential laser phases can be absorbed into the definition of the photon annihilation operator c .

2.1. The no-photon time evolution of the cavity field

Following the ideas of Refs. [12, 13, 14], we assume in the following that the free radiation field surrounding the cavity is in general in its so-called environmentally preferred state, the vacuum state. Denoting the vacuum state by $|0\rangle$ and the state of the cavity at time t by $|\psi_S(t)\rangle$, the total state of the system equals

$$|\psi_{\text{SB}}(t)\rangle = |0\rangle |\psi_S(t)\rangle. \quad (12)$$

Next, we assume that the interaction H_{SB} perturbs $|\psi_{\text{SB}}(t)\rangle$ on a relatively short time scale Δt . As a result, the above state vector evolves into

$$|\psi_{\text{SB}}(t + \Delta t)\rangle = U(t + \Delta t, t) |0\rangle |\psi_S(t)\rangle, \quad (13)$$

where $U(t + \Delta t, t)$ denotes the time evolution operator of the Hamiltonian H . Now we assume that the environment resets the free radiation field rapidly into its initial state. If a measurement reveals that no photon has been created within $(t, t + \Delta t)$, the state of system and bath is given by

$$|0\rangle |\psi_S^0(t + \Delta t)\rangle = |0\rangle \langle 0| U(t + \Delta t, t) |0\rangle |\psi_S(t)\rangle, \quad (14)$$

up to normalisation. Comparing both sides of this equation, we find that

$$|\psi_S^0(t + \Delta t)\rangle = \langle 0| U(t + \Delta t, t) |0\rangle |\psi_S(t)\rangle \quad (15)$$

is the state of the cavity field at time $t + \Delta t$. Changing into an interaction picture with respect to $H_0 = H_S + H_B$ and using second order perturbation theory (as described in Ref. [12] for example) to evaluate the above expression, we find that the state $|\psi_S^0(t)\rangle$ evolves effectively with the non-Hermitian conditional Hamiltonian

$$H_{\text{cond I}} = H_{\text{laser I}} - \frac{i}{2} \hbar \kappa c^\dagger c, \quad (16)$$

under the condition of no photon emission in $(t, t + \Delta t)$ and in the interaction picture. As it should, this Hamiltonian is of the same form as the conditional Hamiltonian in Eq. (6). Comparing both equations, we see that the Lindblad operator L of an optical cavity is given by $L = c$, while the spontaneous decay rate $\Gamma = \kappa$. Finally, let us remark that the probability for the free radiation field to remain in its vacuum state $|0\rangle$ for a time Δt equals

$$P_0(\Delta t) = \| U_{\text{cond I}}(t + \Delta t, t) |\psi_S(t)\rangle \|^2, \quad (17)$$

where $U_{\text{cond I}}(t + \Delta t, t)$ denotes the time evolution operator of the conditional Hamiltonian $H_{\text{cond I}}$.

2.2. Spontaneous photon emission and quantum feedback

Analogously, proceeding as suggested in Refs. [12, 13, 14] and evaluating the density matrix $\rho^\neq(t + \Delta t)$ of an optical cavity under the condition of a photon detection in $(t, t + \Delta t)$ using first order perturbation theory, we find that the state of the resonator immediately after an emission equals [27]

$$|\psi_S^\neq(t + \Delta t)\rangle = \sqrt{\kappa \Delta t} c |\psi_S(t)\rangle, \quad (18)$$

up to normalisation. The normalisation constant of this state squared equals the probability for the emission of a photon in $(t, t + \Delta t)$. Hence

$$I(t) = \kappa \langle c^\dagger c \rangle_t \quad (19)$$

is the probability density for the emission of a photon at time t . Here $\langle c^\dagger c \rangle_t = \langle \psi_S(t) | c^\dagger c | \psi_S(t) \rangle$ denotes the mean number of photons inside the resonator at a time t , when prepared in the state $|\psi_S(t)\rangle$.

If the emission of a photon successfully triggers a feedback pulse, up to normalisation, the state $|\psi_S^\neq(t + \Delta t)\rangle$ of the cavity becomes [25, 27]

$$|\psi_S^\neq(t + \Delta t)\rangle = \sqrt{\kappa} R c |\psi_S(t)\rangle \quad (20)$$

in case of an emission, where R is the unitary operator that describes the effect of the feedback on the resonator field.

2.3. The relevant master equations

To obtain the density matrix $\rho(t)$ of the cavity field, we need to add the density matrix $\rho_S^0(t)$ of the subensemble of cavities with no photon emission in $(t, t + \Delta t)$ and the density matrix $\rho_S^\neq(t)$ of the subensemble of cavities with a photon emission in $(t, t + \Delta t)$. If Δt is sufficiently small, contributions with more than one emission remain negligible and

$$\dot{\rho}_S(t) = \dot{\rho}_S^0(t) + \dot{\rho}_S^\neq(t). \quad (21)$$

For this equation to apply, both density matrices $\rho_S^0(t)$ and $\rho_S^\neq(t)$ need to be normalised such that their relative size coincides with the probability density for an emission or no emission at time t . Taking this into account and using the results of the previous two subsections, we obtain the time derivative $\dot{\rho}_S$ of an optical cavity with continuous laser driving, as shown in Fig. 1. It equals

$$\dot{\rho}_I = -\frac{i}{2}\Omega [c + c^\dagger, \rho_I] + \frac{1}{2}\kappa \left(2c\rho_I c^\dagger - [c^\dagger c, \rho_I]_+ \right) \quad (22)$$

in the interaction picture with respect to the free energy of the resonator field.

Now suppose the continuous laser driving is turned off and a detector monitors the spontaneous leakage of photons through one of its mirrors, as illustrated in Fig. 2. Moreover suppose an instantaneous feedback loop is activated whenever a photon is detected. Proceeding as described above, we find that the master equation of the resonator equals

$$\begin{aligned} \dot{\rho}_I = & \frac{1}{2}\eta\kappa \left(2R_I c\rho_I c^\dagger R_I^\dagger - [c^\dagger c, \rho_I]_+ \right) \\ & + \frac{1}{2}(1 - \eta)\kappa \left(2c\rho_I c^\dagger - [c^\dagger c, \rho_I]_+ \right) \end{aligned} \quad (23)$$

in this case. The first line in Eq. (23) describes resonators that experience feedback, while the terms in the second line take undetected photon emission events into account. As usual, η denotes the detector efficiency. Moreover R_I denotes the unitary operator R in Eq. (20) after transformation into the interaction picture with respect to $H_0 = H_S$. In this paper, we consider quantum feedback in the form of very short and strong resonant laser pulses. This means that R_I is the unitary time evolution operator associated with a laser Hamiltonian with a time-independent Rabi frequency. Hence R_I too is time-independent and equals a displacement operator,

$$R_I = e^{\beta c^\dagger - \beta^* c}, \quad (24)$$

where β is a complex constant that characterises the strength of the feedback pulse. Simplifying Eq. (23) eventually yields

$$\dot{\rho}_I = \frac{1}{2}\kappa \left(2c\rho_I c^\dagger - [c^\dagger c, \rho_I]_+ \right) + \eta\kappa \left(R_I c\rho_I c^\dagger R_I^\dagger - c\rho_I c^\dagger \right). \quad (25)$$

Now we have all the theoretical tools needed to study the dynamics of the experimental setups in Figs. 1 and 2 in more detail.

3. An optical cavity with continuous laser driving

For benchmarking and to later get a better feeling of how quantum feedback alters the dynamics of an open quantum system with spontaneous photon emission, this section analyses the dynamics of an optical cavity with continuous laser-driving from an open systems perspective. After looking at the individual quantum trajectories of the resonator, we derive its stationary state and show that its dynamics are indeed ergodic. We especially emphasise that if the cavity is initially in a coherent state, it remains always coherent.

3.1. Individual quantum trajectories

First we have a closer look at the dynamics of the cavity field under the condition of no photon emission. In the interaction picture with respect to $H_0 = H_S$, the resonator evolves with the conditional time evolution operator

$$U_{\text{cond I}}(t + \Delta t, t) = \exp\left(-\frac{i}{\hbar} H_{\text{cond I}} \Delta t\right) \quad (26)$$

between photon emissions. Using Eq. (16) and applying the Baker-Campbell-Hausdorff formula, one can show that

$$U_{\text{cond I}}(t + \Delta t, t) = e^{-\frac{i}{2}\Omega(c+c^\dagger)\Delta t} e^{-\frac{1}{2}\kappa c^\dagger c \Delta t}, \quad (27)$$

up to terms of second order in Δt . Now suppose the cavity is initially in a coherent state $|\alpha_I(t)\rangle$. In this case, calculating the effect of the second exponential on $|\alpha_I(t)\rangle$ is best done using the Fock basis. To apply the first exponential, we use the general properties of displacement operators with respect to coherent states. Doing so, we find that the state of the cavity at a time $t + \Delta t$ is again a coherent state, which we denote $|\alpha_I(t + \Delta t)\rangle$. To a very good approximation, we find that

$$\alpha_I(t + \Delta t) = e^{-\frac{1}{2}\kappa\Delta t} \alpha_I(t) - \frac{i}{2}\Omega \Delta t. \quad (28)$$

This equation tells us that

$$\dot{\alpha}_I(t) = -\frac{1}{2}\kappa \alpha_I(t) - \frac{i}{2}\Omega, \quad (29)$$

without any approximations. Solving this differential equation for an initial coherent state $|\alpha_I(0)\rangle$, we obtain a general solution for the state $|\alpha_I(t)\rangle$ of the cavity field under the condition of no photon emission in an arbitrarily long time interval $(0, t)$,

$$\alpha_I(t) = e^{-\frac{1}{2}\kappa t} \alpha_I(0) - \frac{i\Omega}{\kappa} \left(1 - e^{-\frac{1}{2}\kappa t}\right). \quad (30)$$

Returning to the Schrödinger picture, the state of the cavity becomes the coherent state $|\alpha_S(t)\rangle$, with $\alpha_S(t)$ given by

$$\alpha_S(t) = \left[e^{-\frac{1}{2}\kappa t} \alpha_S(0) - \frac{i\Omega}{\kappa} \left(1 - e^{-\frac{1}{2}\kappa t}\right) \right] e^{-i\omega_{\text{cav}} t} \quad (31)$$

with $\alpha_S(0) = \alpha_I(0)$. Using Eq. (17), the calculation that lead to Eq. (28) moreover reveals that

$$P_0(\Delta t) = \exp\left[-|\alpha_S(t)|^2 (1 - e^{-\kappa\Delta t})\right] \quad (32)$$

is the probability for no photon emission in a short time interval $(t, t + \Delta t)$.

Finally, we have a closer look at the effect of a photon emission on the state of the resonator field. Using Eq. (18), we immediately see that the spontaneous emission of a photon does not change the field inside the resonator, if the cavity is initially in a coherent state, since coherent states are the eigenstates of the photon annihilation operator c . Hence, Eq. (31) not only describes the state of the cavity field under the condition of no photon emission in the time interval $(0, t)$, but also describes the state of the resonator in case of emissions.

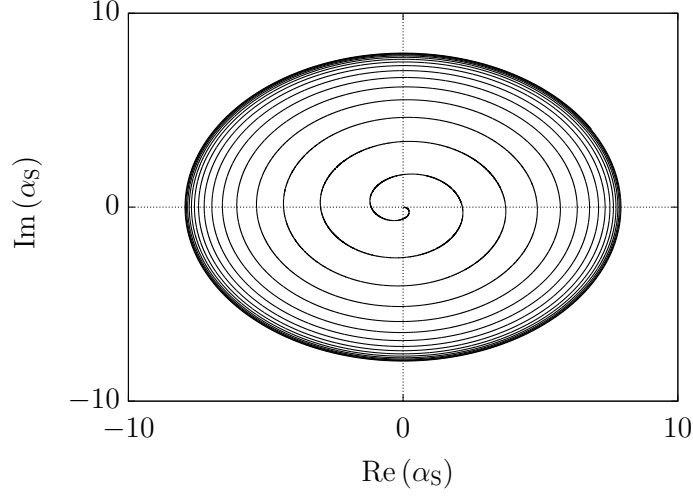


Figure 3. [Colour online] Phase space diagram illustrating the dynamics of a laser-driven optical cavity with spontaneous cavity decay in the Schrödinger picture. The cavity is initially in its vacuum state $|0\rangle$ and $\Omega = 8\kappa$. The field inside the resonator remains always in the coherent state $|\alpha_S(t)\rangle$ given in Eq. (31). The state of the cavity follows an outwards spiral until it eventually reaches the stable circular orbit described by Eq. (33).

3.2. Long-term dynamics

Fig. 3 visualises the trajectory of an optical cavity with continuous laser driving with the help of a phase diagram. This is possible since the resonator remains always in a coherent state $|\alpha_S\rangle$, which can be represented by a point in the complex plane. The x -co-ordinate of this point equals the real part of α_S , while the y -co-ordinate coincides with its imaginary part. Suppose the resonator is initially in its vacuum state with $\alpha_S(0) = 0$. Then $\alpha_S(t)$ in the Schrödinger picture describes an outwards spiral, which starts at the origin and eventually reaches a stable circular orbit. Eventually, the distance of the complex numbers $\alpha_S(t)$ from the origin remains constant in time. Eq. (31) shows that

$$\alpha_S(t) = -\frac{i\Omega}{\kappa} e^{-i\omega_{\text{cav}} t} \quad (33)$$

to a very good approximation, when t is sufficiently large. In the interaction picture (c.f. Eq. (30)), $\alpha_I(t)$ assumes the stationary state value $\alpha_{\text{ss I}}$,

$$\alpha_{\text{ss I}} = -\frac{i\Omega}{\kappa}. \quad (34)$$

This state is invariant under the no-photon time evolution of the system as well as being immune to the spontaneous emission of a photon.

Moreover, Eq. (19) in the previous subsection allows us to calculate the probability density $I(t)$ for the emission of a photon at any time t . If the system is initially prepared in a coherent state $|\alpha_S(0)\rangle$, it remains coherent and $I(t)$ simply equals

$$I(t) = \kappa |\alpha_S(t)|^2. \quad (35)$$

Substituting Eq. (31) into this equation yields

$$\begin{aligned} I(t) = & \kappa e^{-\kappa t} |\alpha_S(0)|^2 + \frac{\Omega^2}{\kappa} \left(1 - e^{-\frac{1}{2}\kappa t}\right)^2 \\ & - 2\Omega e^{-\frac{1}{2}\kappa t} \left(1 - e^{-\frac{1}{2}\kappa t}\right) \text{Im}(\alpha_S(0)). \end{aligned} \quad (36)$$

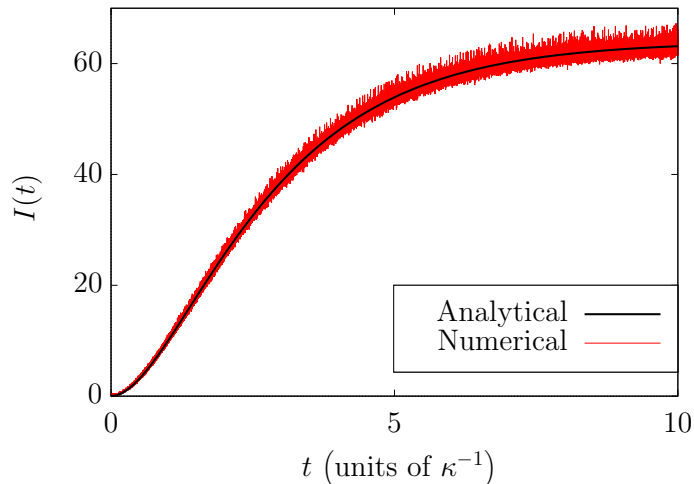


Figure 4. [Colour online] Time dependence of the photon emission rate $I(t)$ of a laser-driven optical cavity. As in Fig. 3, the resonator is initially in its vacuum state and $\Omega = 8\kappa$. The emission rate $I(t)$ soon reaches its stationary state value I_{ss} in Eq. (37). A comparison between the analytical result in Eq. (36) and a quantum jump simulation of $I(t)$, which averages over 10^6 individual quantum trajectories, shows relatively good agreement.

Fig. 4 shows $I(t)$ for the case where the cavity is initially in its vacuum state and $I(0) = 0$. Eventually, $I(t)$ assumes the constant value I_{ss} ,

$$I_{ss} = \frac{\Omega^2}{\kappa}, \quad (37)$$

which is the stationary state photon emission rate of the laser-driven optical cavity. To obtain numerical results, we determine the mean number of photon emissions within short time intervals $(t, t + \Delta t)$ by averaging over a large number of quantum trajectories. We observe relatively good agreement between analytical and numerical results. This agreement increases when more individual quantum trajectories are taken into account.

3.3. Ergodicity

To decide whether the dynamics of the cavity field are ergodic or non-ergodic, we need to check whether we can deduce all statistical properties of this physical system from a single, sufficiently long quantum trajectory. In the previous two subsections, we have seen that the state of the resonator $|\alpha_I(t)\rangle$ at any time t is the same for all possible quantum trajectories, no matter when photon emissions occur. Moreover, we have seen that $\alpha_I(t)$ reaches its stationary state value α_{ss} in Eq. (34) relatively rapidly. It is therefore not surprising to find that the experimental setup in Fig. 1 is indeed ergodic. The time and ensemble averages of a laser driven optical cavity with spontaneous photon emission are the same.

4. An optical cavity inside an instantaneous quantum feedback loop

In this section, we replace the continuous laser-driving of the resonator with instantaneous quantum feedback. As illustrated in Fig. 2, we assume that the spontaneous emission of a photon triggers a short, strong resonant laser pulse. For simplicity, we assume that the effect of the laser is instantaneous, which applies to a very good approximation if its length is short compared to the inverse of the cavity decay rate κ . In the following, we see that changing the way in which energy is fed into the system alters the system dynamics dramatically.

4.1. Individual quantum trajectories

As in Section 3, we first have a closer look at the individual quantum trajectories of the resonator. As before, we model the time evolution between photon emissions by a non-Hermitian conditional Hamiltonian. Without laser driving, $\Omega = 0$ and H_{cond} in Eq. (16) simplifies to a single damping term. From Eq. (30) we see that the cavity evolves in the interaction picture into the state $|\alpha_I(t)\rangle$ with

$$\alpha_I(t) = e^{-\frac{1}{2}\kappa t} \alpha_I(0) \quad (38)$$

in this case, if it is initially prepared in the coherent state $|\alpha_I(0)\rangle$. The probability $P_0(\Delta t)$ for no photon emission in a short time interval $(t, t + \Delta t)$ is still given by Eq. (32), which is later taken into account when we simulate quantum trajectories.

Next we have a closer look at the effect of a spontaneous photon emission. Again the state of the cavity remains unchanged, if the emission of a photon goes unnoticed. However, if the emission of a photon is detected and successfully triggers a feedback pulse, the state of the cavity changes from $|\alpha_I(t)\rangle$ into $|\alpha_I(t + \Delta t)\rangle = R_I |\alpha_I(t)\rangle$, with $R_I(t)$ given in Eq. (24). Using the properties of displacement operators we find

$$|\alpha_I(t + \Delta t)\rangle = |\alpha_I(t) + \beta\rangle, \quad (39)$$

given that a feedback pulse is applied at a time t . The state of the resonator is no longer invariant under photon emission. Furthermore, even when all parameters are kept the same, every realisation of the described process now results in a different quantum trajectory.

4.2. Non-ergodicity

The phase space diagrams in Fig. 5 show random samples of individual quantum trajectories in the Schrödinger picture. Figs. 5(a)–(c) and Figs. 5(d)–(f), respectively, depict the same ten runs of the experiment. In Fig. 5(a)–(c) we have $\alpha_S(0) = 2$ and in Fig. 5(d)–(f) we have $\alpha_S(0) = -2$, while the feedback pulse in both cases is given by $\beta = 2$. There is a gradual zoom from (a) to (c) and from (d) to (f) to clearly show the difference between the dynamics of the resonator field for relatively large and for relatively small mean photon numbers. Most importantly, there are now two different types of dynamics. Many of the shown quantum trajectories move further and further away from the vacuum state. Once the amount of excitation inside the cavity reaches a certain threshold, the mean number of photons inside the resonator is likely to keep increasing. For other trajectories, the mean number of photons inside the resonator remains relatively small. Those trajectories are extremely likely to eventually reach the vacuum state, where the cavity cannot emit another photon and the dynamics of the system comes to a hold.

Moreover, Fig. 6 shows the time-dependence of the amplitude of $\alpha_S(t)$ for the same runs of the experiment as in Fig. 5. Now the existence of two different types of dynamics becomes even more evident. Either the amount of excitation inside the resonator grows very rapidly in time or the cavity field approaches its vacuum state with $|\alpha_S| = 0$. This is not surprising. On one hand, photon emissions at a relatively high rate attract more feedback pulses, thereby further increasing the amount of excitation inside the resonator. On the other hand, in the absence of any photon emissions the no-photon time evolution with the conditional Hamiltonian H_{cond} continuously reduces the field amplitude $|\alpha_S(t)|$ (c.f. Eq. (38)). This is due to the fact that not emitting a photon gradually reveals information about the quantum state of the resonator, which then needs to be updated accordingly [12]. Due to the fact that the cavity randomly exhibits two different types of dynamics, the statistical properties of the resonator field can no longer be deduced from a single, sufficiently long run of a single experiment. Consequently, the dynamics of an optical cavity inside an instantaneous feedback loop are non-ergodic.

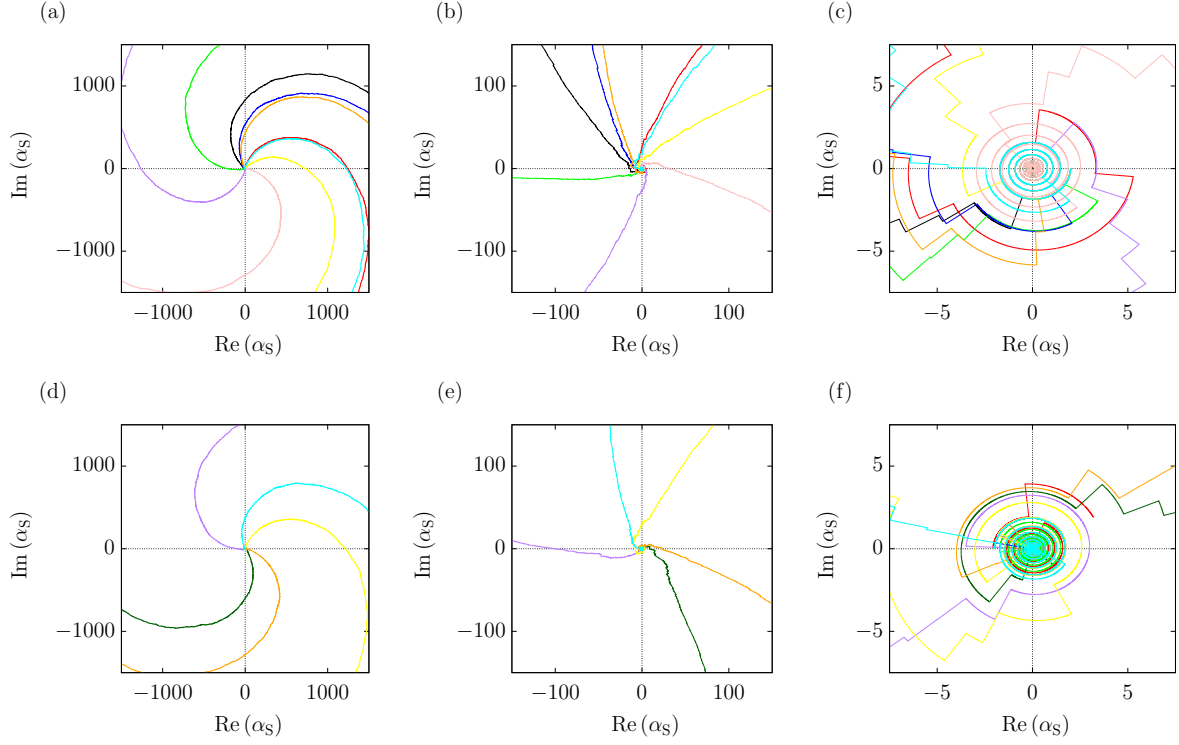


Figure 5. [Colour online] Phase space diagrams illustrating the dynamics of a random sample of ten possible quantum trajectories of an optical cavity inside an instantaneous quantum feedback loop with $\beta = 2$ and $\eta = 0.5$. All trajectories are the result of a quantum jump simulation of length $t = 10\kappa^{-1}$. In (a)–(c), we consider an initial state $|\alpha_S(0)\rangle$ with $\alpha_S(0) = 2$. One trajectory eventually reaches the vacuum state, while all others move further and further away from the origin. In (d)–(f), we have $\alpha_S(0) = -2$. Now we see that only five of the ten trajectories diverge, while the other five appear to be converging. The diagrams in every row only differ by the size of the phase space volume, which is shown.

How likely it is for a single quantum trajectory to exhibit a certain type of behaviour depends strongly on the initial state of the resonator. This is illustrated in Fig. 7, which shows the probability for an individual quantum trajectory to eventually reach the vacuum state as a function of the initial state $|\alpha_S(0)\rangle$. For example, if the cavity is initially in $|0\rangle$, Fig. 7 shows that it remains there with unit probability. However, when moving the initial state of the cavity field away from the vacuum, it becomes more and more likely that the resonator keeps on accumulating photon excitations. For sufficiently large values of $\alpha_S(0)$, effectively all the possible trajectories of the cavity field diverge. As we shall see below, the vacuum can become an repulsive fixed point of the system dynamics, even when averaged over a large number of repetitions of the same experiment.

4.3. Non-linear dynamics of ensemble averages

In this subsection, we have a closer look at the time-dependence of expectation value averaged over a large ensemble of individual quantum trajectories. Since the cavity field remains always in a coherent state, the density matrix $\rho_I(t)$, which allows us to calculate the expectation value $\langle A \rangle_t = \text{Tr}(A\rho_I(t))$ of an observable A at any time t , needs to be a statistical mixture of coherent

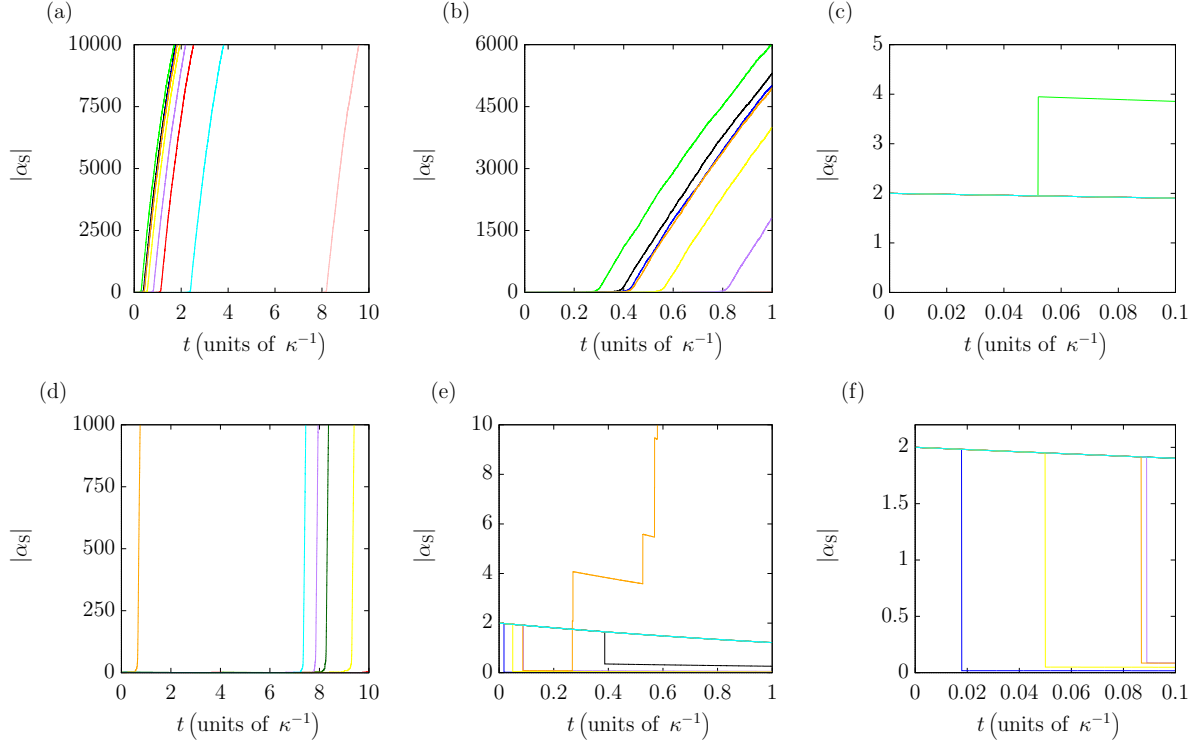


Figure 6. [Colour online] Explicit time dependence of the amplitude $|\alpha_S(t)|$ of the states $|\alpha_S(t)\rangle$ of the individual quantum trajectories shown in Fig. 5. These diagrams illustrate even more clearly than Fig. 5 that there are two types of dynamics. Either the mean number of photons inside the cavity increases in time and keeps on growing or the electric field amplitudes $|\alpha_S(t)|$ eventually becomes relatively small.

states. Hence, without restrictions we can assume that

$$\rho_I(t) = \int_{\mathcal{C}} d\alpha P(\alpha, t) |\alpha\rangle\langle\alpha|, \quad (40)$$

where the $P(\alpha, t)$ denote time-dependent probabilities. Taking this into account, using Eqs. (24) and (25) and calculating the density matrix ρ_I with $\dot{\rho}_I = 0$, we find that the cavity possesses a unique stationary state ρ_{ss} . As one would expect, this state is the vacuum state of the resonator,

$$\rho_{ss} = |0\rangle\langle 0|. \quad (41)$$

As we have seen already in the previous subsection, when reaching this state the dynamics of the resonator come to a hold. However, Fig. 7 shows that often the cavity does not reach this state. For a wide range of initial states $|\alpha_S(0)\rangle$, there is a significant probability for the mean number of photons inside the cavity to continue to grow in time.

To calculate the probability density $I(t)$ for the emission of a photon at time t averaged over a large ensemble of individual quantum trajectories, we now have a closer look at the dynamics of the photon number operator. Using the master equation (25), one can show that

$$\langle \dot{A} \rangle_t = \frac{1}{2} \kappa \left\langle 2 c^\dagger A c - [A, c^\dagger c]_+ \right\rangle_t + \eta \kappa \left\langle c^\dagger R_I^\dagger A R_I c - c^\dagger A c \right\rangle_t \quad (42)$$

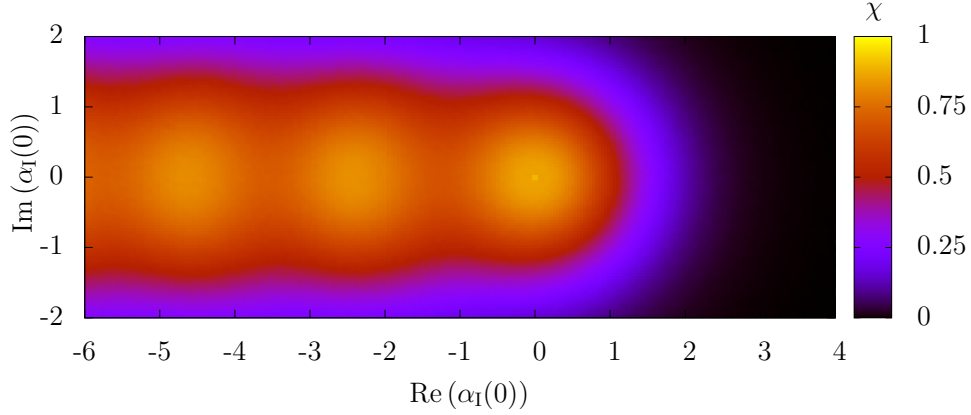


Figure 7. [Colour online] Heat-plot showing the dependence of the probability χ on the initial coherent state $|\alpha_I(0)\rangle$ of the resonator. Here χ denotes the probability that coherent state $|\alpha_I(t)\rangle$ of the cavity is such that $|\alpha_I(t)| < 0.1$ after a time $t = 10\kappa^{-1}$. Hence it equals the probability that the cavity eventually reaches its vacuum state to a very good approximation. The plot is the result of a quantum jump simulation which averages over 10^4 quantum trajectories. As in Figs. 5 and 6, we assume $\beta = 2$ and $\eta = 0.5$.

applies for the expectation value of any observable A . Setting $A = c^\dagger c$ and taking into account that $I(t) = \kappa \langle c^\dagger c \rangle_t$ and $[c, c^\dagger] = 1$, the above differential equation yields

$$\dot{I}(t) = -\kappa^2 \int_{\mathcal{D}} d\alpha [1 - \eta(|\alpha + \beta|^2 - |\alpha|^2)] P(\alpha, t) |\alpha|^2 \quad (43)$$

for the density matrix $\rho_I(t)$ in Eq. (40). In the absence of any feedback, i.e. for $\eta = 0$, Eq. (43) simplifies to the simple linear differential equation $\dot{I}(t) = -\kappa I(t)$. However, in the presence of sufficiently strong feedback, the η -term dominates the dynamics of $I(t)$ and makes it non-linear. For relatively large values of η and β , $\dot{I}(t)$ even becomes positive and the mean number of photons inside the resonator grows in time. As we have seen in Ref. [27], the phase space volume occupied by the resonator grows in time in case of sufficiently strong instantaneous quantum feedback, which is often the case for non-ergodic systems.

4.4. Instability of the stationary state

The aforementioned dynamics are in stark contrast to other quantum optical systems with spontaneous photon emission. Like the experimental setup which we studied in Section 3, quantum optical systems usually occupy a constantly shrinking phase space volume. Eventually they reach a stationary state that is independent of their initial state. In other words, the stationary state of a quantum optical system is usually an attractive fixed point of its dynamics. In contrast to this, as we shall see below, the stationary state of an optical cavity inside a quantum feedback loop becomes repulsive if the feedback is sufficiently strong.

Suppose the cavity is initially in its vacuum state and a small perturbation moves the resonator into a coherent state $|\alpha_I\rangle$ with $|\alpha_I|^2 \ll |\beta|^2$. When this applies, Eq. (43) simplifies to

$$\dot{I}(t) = -\kappa^2 (1 - \eta|\beta|^2) |\alpha_I|^2. \quad (44)$$

The right hand side of this equations becomes positive when

$$\eta|\beta|^2 > 1. \quad (45)$$

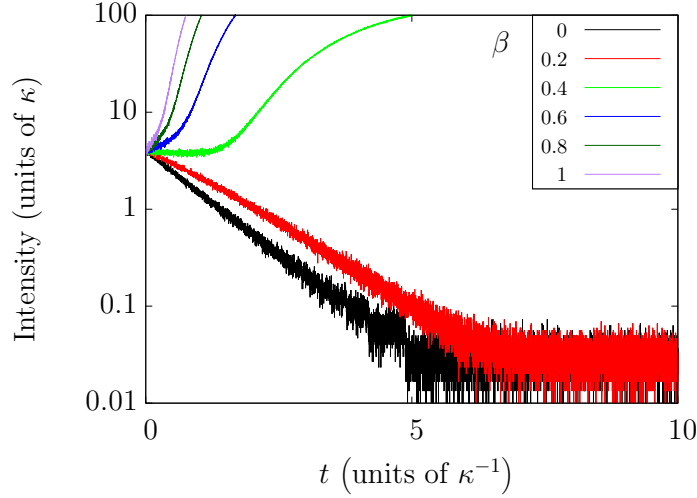


Figure 8. [Colour online] Time dependence of the photon emission rate $I(t)$ for different feedback parameters β . As in previous figures, we assume $\alpha_I(0) = 2$ and $\eta = 0.5$. The figure is again the result of a quantum jump simulation, which averages over 10^6 individual quantum trajectories. For relatively small values of β , $I(t)$ tends to zero. However, as β increases, the dynamics of the cavity change and the mean number of photons inside the resonator continues to grow in time.

This shows that the ensemble average of the mean number of photons inside the resonator increases further in time in case of sufficiently strong feedback. The stationary state of the resonator is no longer an attractor of the system dynamics but a repulsive fixed point. This behaviour is illustrated in Fig. 8, which shows the time dependence of the mean cavity photon emission rate $I(t)$ for quantum feedback pulses of different strength, i.e. for different values of β . As expected, the photon emission rate $I(t)$ only converges to zero when β is relatively small.

4.5. Persistent dependence of ensemble averages on initial states

Perhaps even more surprising than the instability of the stationary state is the fact that the dynamics of the ensemble averages of expectation values depend strongly on the initial coherent state $|\alpha_S(0)\rangle$ of the cavity. In contrast to many other quantum optical systems, information about the initial state of the quantum system is never lost. This behaviour is illustrated in Fig. 9, which shows the time-dependence of the photon emission rate $I(t)$ of the resonator for different initial states $|\alpha_S(0)\rangle$ with $\alpha_S(0) = |\alpha_S(0)|e^{i\varphi}$. The amplitude of $|\alpha_S(0)|$ and the feedback parameter β are the same in each case, but the considered phases φ differ for different curves. The figure clearly illustrates that information about φ is not lost. The emission rate $I(t)$ does not converge to a single value. On the contrary, the distance between curves that correspond to different values of φ even increases in time. This interesting and highly unusual feature of open quantum systems opens the way to novel applications. For example, we recently showed that an optical cavity inside an instantaneous quantum feedback loop can be used to measure the phase shift between two pathways of light with an accuracy that exceeds the standard quantum limit [27].

5. Conclusions

This paper addresses the question of how the often relatively complex dynamics of classical systems might arise in open quantum systems. For example, individual trajectories of

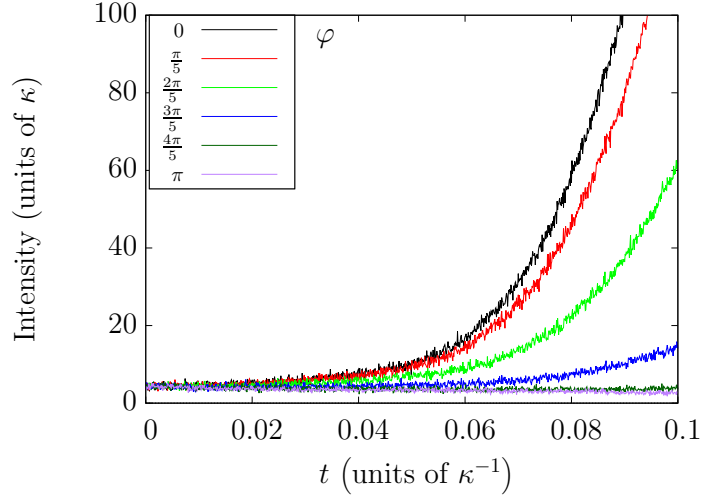


Figure 9. [Colour online] Time dependence of the photon emission rate $I(t)$ for different initial states $|\alpha_S(0)\rangle$ with varying phase φ . Here $|\alpha_S(0)| = 2$, $\beta = 2$ and $\eta = 0.5$. The figure is again the result of a quantum jump simulation which averages over 10^6 possible trajectories. The figure illustrates that there is a strong dependence of the dynamics of ensemble averages on the initial state of the resonator.

open quantum systems with Markovian dynamics are usually ergodic and rapidly lose their dependence on their respective initial state [9]. In contrast to this, classical systems often evolve according to a set of non-linear differential equations. Classical stochastic processes are often non-ergodic and even chaotic, whereas classical systems usually maintain a dependence on initial conditions [1, 7, 8]. Nevertheless, it is widely believed that classical dynamics emerges from the behaviour of microscopic quantum systems. Quantum physics is believed to underly all other less complex physical theories. However, less is known about the mechanisms which induce classicality. The search for such mechanisms is an active area of research [5, 6, 19, 20, 21, 22].

This paper identifies quantum feedback as a tool that dramatically alters the dynamics of open quantum systems. As an example, we study the stochastic dynamics of the electromagnetic field of an optical cavity inside an instantaneous quantum feedback loop. It is shown that the dynamics of ensemble averages, like the mean number of photons inside the resonator, can become non-linear even in the absence of non-linear interactions. In the presence of sufficiently strong feedback, the only fixed point of the dynamics of the cavity, i.e. its unique stationary state, can become repulsive. Moreover, it is no longer possible to deduce the the dynamics of ensemble averages from individual quantum trajectories, which implies non-ergodicity. The dynamics of ensemble averages can depend strongly on the initial state of the cavity field. This feature can be employed, for example, in quantum-enhanced metrology [27]. In summary, the dynamics of even relatively simple quantum systems can become much more complex in the presence of back actions from the surrounding environment.

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not require supporting research data.

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